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(Super)gravity and Yang-Mills Theories as Generalized Topological Fields with Constraints

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Abstract

We present a general approach to construct a class of generalized topological field theories with constraints by means of generalized differential calculus and its application to connection theory. It turns out that not only the ordinary BF formulations of general relativity and Yang-Mills theories, but also the $N = 1, 2$ chiral supergravities can be reformulated as these constrained generalized topological field theories once the free parameters in the Lagrangian are specially chosen. We also show that the Chern-Simons action on the boundary may naturally be induced from the generalized topological action in the bulk, rather than introduced by hand.

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I. INTRODUCTION

Recent progress in the study of quantum theory of gravity in terms of Ashtekar-Sen variables [1] has revealed that incorporating the methods of topological quantum field theory into non-perturbative quantum gravity may push this approach forward significantly (for a recent review, see, e.g. [2]). The starting point is that the classical Lagrangian of gravity theories in the first order formalism can always be written as constrained topological field theories. Such an extraordinary example is the Plebanski-like Lagrangian for general relativity (GR) with cosmological constant [3]. It has the form of BF theory with appropriate gauge group in four dimensions

$$S_{BF}(A,B)=\int_M Tr(B\wedge F+\frac{\Lambda}{12}B\wedge B). \tag{1}$$

As a matter of fact, such a framework can be extended to construct (at least $N\leq 2$) supergravities in fours dimensions [4], as well as $D=11$ supergravity [5].

Very recently, we have shown [6] that based on the Generalized Differential Calculus (GDC) [7, 8](see also the appendix), a generalized connection theory may be established. Furthermore, Einstein's general relativity can be reformulated from a kind of generalized topological field theory ($GTFT$). The GDC originally was advocated to study the twistor theory. In this context an ordinary p-form on a manifold M is extended to a generalized p-form defined by an ordered pair of the original p-form and an additional $(p+1)$-form on the same manifold. Correspondingly, the ordinary exterior differential d is replaced by the generalized exterior differential $\mathbf{d}$ that is also nilpotent. Thus when applying this GDC to the principle bundle $P(M,G)$ on M with gauge group G, the ordinary g-valued connection 1-form A and curvature 2-form F are generalized to a g-valued generalized connection 1-form $\mathbf{A}$ and a curvature 2-form $\mathbf{F}$, respectively. By definition, $\mathbf{A}$ is a g-valued pairing of (A,B) with B an ordinary g-valued gauge covariant 2-form, while $\mathbf{F}$ is the pairing $(F+kB, D_AB)$ with k an arbitrary constant and D_A the ordinary covariant derivative with respect to A. It has also been shown in [6] that with respect to $\mathbf{d}$, $\mathbf{F}$ satisfies the Bianchi identity and its gauge invariant polynomials are endowed with the Chern-Weil homomorphism. Since there is an arbitrary gauge covariant 2-form B in the definition of $\mathbf{A}$, this framework may combine an ordinary topological term in topological field theory (TFT) with other terms as the parts in $GTFT$ with particular chosen field B. In addition, it is interesting to see that this kind

of $GTFT$ also leads to certain relation between the ordinary local Chern-Simoms term on the boundary ∂M and the BF-Lagrangian in the bulk M as was shown in [6].

In this paper, we intend to develop a general approach to reformulate other gravity theories, particularly the $N = 1, 2$ chiral supergravities, as well as pure Yang-Mills theory together with corresponding topological term, respectively, as this sort of the constrained $GTFTs$. Namely, we show that the ordinary Lagrangian of gravity theories together with ordinary topological terms may be simply written as a *generalized* topological term with appropriate constraints. This generalized topological action has a form of the integral of the generalized second Chern class as Lagrangian on four dimensions

$$S = \int_M STr(\mathbf{F} \wedge \mathbf{F}), \tag{2}$$

where STr is understood as the supertrace in the case of supergravities. This action includes both the ordinary (topological) second Chern class $STr(F \wedge F)$ and BF-type terms as well as their super-partners in the case of supergravities. Thus, this offers a framework of $GTFT$ with constraints for specially chosen B or B^s in the case of gravity and Yang-Mills or supergravities, respectively. It is worthwhile to mention that this formalism provides a very convenient framework for constructing the ($N = 1, 2$) supergravities, since it combines all components of the theory into a compact manner so as to the gauged Lie super-algebra is automatically closed.

The paper is arranged as follows. In section two we present the general formalism of $GTFT$ based on the generalized (super)-connection theory. Its topological features may be formally understood by considering the generalized Chern-Weil homomorphism for generalized curvature invariant polynomials. Then we exhibit the geometric formulation of Einstein-Hilbert action and the Yang-Mills theory on curved spacetime as this kind of $GTFTs$ with constraints in section three. In section four and five, we turn to show that $N = 1$ and $N = 2$ chiral supergravities with some free parameters can also be formulated as the $GTFT$ with constraints, respectively. In all these cases, we find that the BF-type action in the bulk is always as a counterpart of the ordinary topological terms, which may naturally induce Chern-Simons-type action on the boundary ∂M. We end this paper by some remarks focusing on its possible applications to the quantization of gravitational field, which should be further considered in the future.

II. GENERAL FORMALISM FOR THE GENERALIZED TOPOLOGICAL FIELD THEORY

In this section, we first recall the properties of the generalized gauge fields [6] in the framework of GDC [6–8]. Then we generalize it to the supersymmetric case. Namely, we introduce the Lie super-algebra g^s valued gauge fields as well as their gauge invariant generalized curvature polynomials.

We adopt the conventions and notations through this paper as follows. The Lie super-algebra g^s-valued connections and curvatures are written as script letters, while the generalized super-connection and super-curvature in the context of GDC are written as the bold letters. The upper-case Latin letters $A, B, ... = 0, 1$ denote two component spinor indices, which are raised and lowered with the constant symplectic spinors $\epsilon_{AB} = -\epsilon_{BA}$ together with its inverse and their conjugates according to the conventions $\epsilon_{01} = \epsilon^{01} = +1$, $\lambda^A := \epsilon^{AB}\lambda_B$, $\mu_B := \mu^A\epsilon_{AB}$ [9].

Let us consider a principle bundle $P(M^4, G)$ over the 4-dimensional manifold M^4 with a semisimple gauge group G as the structure group. A Lie algebra g-valued generalized connection $\mathbf{A}$ is defined as a g-valued pairing of a g-valued ordinary connection 1-form A and a g-valued ordinary 2-form B

$$\mathbf{A} = (A^p, B^p)T_p = (A, B), \quad T_p \in g, \tag{3}$$

where B is assumed as gauge covariant under the gauge transformations in order to introduce a g-valued generalized curvature 2-form $\mathbf{F}$ that is gauge covariant

$$\begin{aligned} \mathbf{F} &= \mathbf{dA} + \mathbf{A} \wedge \mathbf{A} \\ &= (dA + A \wedge A + kB, \quad dB + A \wedge B - B \wedge A) \\ &= (F + kB,\ DB), \end{aligned} \tag{4}$$

where $D = D_A$ is the covariant derivative with respect to the connect A, k is an arbitrary constant. It is easy to show that the g-valued generalized curvature 2-form $\mathbf{F}$ satisfies the Bianchi identity via GDC:

$$\begin{aligned} \mathbf{DF} &:= \mathbf{dF} + \mathbf{A} \wedge \mathbf{F} - \mathbf{F} \wedge \mathbf{A} \\ &= (DF,\ D^2B) \equiv 0. \end{aligned} \tag{5}$$

In [6], we establish the generalized Chern-Weil homomorphism for generalized curvature invariant polynomials via GDC on any even dimensional manifolds. But their topological meaning should be as same as the original curvature invariant polynomials. The reason is that the cohomology with respect to $\mathbf{d}$ in GDC is trivial, which is easy to be proved (see the appendix).

Now we may introduce an action on the base manifold M^4 as the topological form with the generalized second Chern-class as Lagrangian for the generalized curvature 2-form:

$$\mathcal{S}_T = \int_{M^4} \mathcal{L}_T = \int_{M^4} Tr(\mathbf{F} \wedge \mathbf{F}) = \int_{M^4} \mathbf{dQ}_{CS}. \tag{6}$$

Here $\mathbf{Q}_{CS}$ is the generalized local Chern-Simons 3-form, i.e., the pairing of a 3-form and a 4-form

$$\begin{aligned} \mathbf{Q}_{CS} &= Tr(\mathbf{A} \wedge \mathbf{F} - \tfrac{1}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}) \\ &= Tr(A \wedge F - \tfrac{1}{3} A \wedge A \wedge A + kA \wedge B, \\ &\qquad A \wedge DB + B \wedge F + kB \wedge B). \end{aligned} \tag{7}$$

Note that in the pairing of the local generalized Chern-Simons term above, the 3-form is the usual Chern-Simons term up to an $k\,Tr(A \wedge B)$ term while the 4-form is the usual BF term up to a $Tr(A \wedge DB)$ term. Namely, although the topological meaning of the generalized second Chern class is the same as before, the generalized local Chern-Simons term combines the ordinary Chern-Simons 3-form and the BF 4-form together as an entire object via GDC. This already leads to a relation between the Chern-Simons term and the BF-term.

In fact, the generalized second Chern class as a Lagrangian 4-form in (6) is a pairing of a 4-form and a 5-form:

$$\begin{aligned} \mathcal{L}_{\mathrm{T}} &= Tr(F \wedge F + 2kB \wedge F + k^2 B \wedge B, \\ &\qquad 2(F \wedge DB + kB \wedge DB)). \end{aligned}$$

Using the Bianchi identity, we can rearrange the 5-form so that

$$\begin{aligned} \mathcal{L}_{\mathrm{T}} &= Tr(F \wedge F + 2kB \wedge F + k^2 B \wedge B, \\ &\qquad d\,(2B \wedge F + kB \wedge B)). \end{aligned} \tag{8}$$

The first term is just the ordinary topological term, i.e. the second Chern class $Tr(F \wedge F)$, together with BF Lagrangian as its counterpart appeared via GDC. And the second term is a total derivative of the BF Lagrangian.

Once more, the pairing of the action (6) shows a relation between two types of $TFTs$, the Chern-Simons-type on the boundary and the BF-type in the bulk on four dimensions:

$$\begin{aligned}\mathcal{S}_{\mathrm{T}}[\mathbf{A}] &= \int_{M^4} \mathcal{L}_T = \int_{M^4} \mathbf{dQ}_{CS} \\ &= \int_{M^4} Tr\left(F \wedge F + 2kB \wedge F + k^2 B \wedge B\right). \end{aligned} \tag{9}$$

In addition, although as a five form, the total derivative of the BF Lagrangian in (8) vanishes automatically on the 4-dimensional manifold M^4, it also indicates that the BF Lagrangian on four dimensions may be regarded as a term on the boundary of a topological action (6) on a five dimensional buck. This implies that there may exist a kind of decent relations among these topological terms on different dimensions. We will leave this topic for forthcoming publications.

Let us now consider the supersymmetric generalization of the above issues. In fact, if we still consider the principle bundle $P(M^4, G^s)$ with a Lie super-group G^s as the structure group, it is more or less straightforward to generalize the above issues to the supersymmetric case as long as the Lie algebra g is replaced by a Lie super-algebra g^s of G^s.

For the sake of explicitness, we may take the Lie super-algebra $Osp(2|N)$ as an example. The ordinary super connection 1-form valued on this Lie super-algebra is defined as,

$$\mathcal{A} = A^{AB} J_{AB} + \psi^{IA} Q_{IA} + A^{IJ} Z_{IJ}, \tag{10}$$

where J_{AB}, Q_{IA} are the bosonic, fermionic generators of the algebra, respectively, and Z_{IJ} the generators of automorphism group $SO(N)$. The indices IJ are antisymmetric and may run from 1 to N, depending on the order of supersymmetry.

In order to apply the GDC to the case of super-algebra, we introduce an ordinary 2-form fields valued on g^s, which should be gauge covariant,

$$\mathcal{B} = B^{AB} J_{AB} + B^{IA} Q_{IA} + B^{IJ} Z_{IJ}, \tag{11}$$

and define a generalized connection 1-form as

$$\mathbf{A}^{\mathbf{s}} = (\mathcal{A}, \mathcal{B}) = (A^{AB}, B^{AB}) J_{AB} + (\psi^{IA}, B^{IA}) Q_{IA} + (A^{IJ}, B^{IJ}) Z_{IJ}, \tag{12}$$

where the upper index s denotes that the object is a Lie super-algebra valued. In what follows, it is often omitted for the sake of simplicity.

Following the rules of GDC, we find that the Lie super-algebra valued generalized curvature 2-form may be defined as

$$\begin{aligned}
\mathbf{F} &= \mathbf{dA} + \mathbf{A} \wedge \mathbf{A} \\
&= (F^{AB} + kB^{AB} - \frac{a}{2}\psi^{IA} \wedge \psi_I^B, DB^{AB} - a\psi^{IA} \wedge B_I^B)J_{AB} \\
&\quad +(F^{IA} + kB^{IA} + \frac{a}{2}A^{IJ} \wedge \psi_J^A, DB^{IA} + aA^{IJ} \wedge B_J^A)Q_{IA} \\
&\quad +(F^{IJ} + kB^{IJ} + \frac{1}{2}\psi^{IA} \wedge \psi_A^J, DB^{IJ} + \psi^{IA} \wedge B_A^J)Z_{IJ},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
F^{AB} &= dA^{AB} + A^{AC} \wedge A_C{}^B, \qquad F^{IA} = d\psi^{IA} + A^{AB} \wedge \psi_B^I, \\
F^{IJ} &= dA^{IJ} + A^{IK} \wedge A_K^J,
\end{aligned} \tag{14}$$

and a is some coupling constant appearing in the superalgebra. It is straightforward to show that this Lie super-algebra valued generalized curvature satisfies the Bianchi identity:

$$\begin{aligned}
\mathbf{DF} &:= \mathbf{dF} + \mathbf{A} \wedge \mathbf{F} - \mathbf{F} \wedge \mathbf{A} \\
&= (DF^{AB}, D^2B^{AB})J_{AB} + (DF^{IA}, D^2B^{IA})Q_{IA} + (\tilde{D}F^{IJ}, \tilde{D}^2B^{IJ})Z_{IJ} = 0.
\end{aligned} \tag{15}$$

Now it is also easy to show that the formal topological meaning for the gauge invariant polynomial, $\mathcal{P}(\mathbf{F})$, of the Lie super-algebra valued generalized curvature 2-forms. Namely, it is closed with respect to $\mathbf{d}$ and satisfies the Chern-Weil homomorphism:

$$(i) \qquad \mathbf{d}\mathcal{P}(\mathbf{F}) = 0, \tag{16}$$

$$(ii) \qquad \mathcal{P}(\mathbf{F}_1) - \mathcal{P}(\mathbf{F}_0) = \mathbf{d}\mathcal{Q}(\mathbf{A}_0, \mathbf{A}_1), \tag{17}$$

where $\mathcal{Q}(\mathbf{A}_0, \mathbf{A}_1)$ is the secondary Chern-Simons invariant polynomial for the generalized curvature.

Now, we are ready to introduce the action as the integral of the generalized second Chern class:

$$S = \int_M STr\mathbf{F} \wedge \mathbf{F}. \tag{18}$$

The formal topological character of the action (18) can be understood by investigating the above Chern-Weil homomorphism formula as we did in [6]. In particular, the Lagrangian 4-form $\mathcal{L}$ can be given by taking $\mathcal{P}(\mathbf{F})$ as the second Chern class for the Lie super-algebra valued generalized curvature, as well as $\mathbf{A}_1 = \mathbf{A}$ and $\mathbf{A}_0 = 0$ in (17), then

$$STr(\mathbf{F} \wedge \mathbf{F}) = \mathbf{d}\mathcal{Q}_{CS}, \tag{19}$$

where $\mathcal{Q}_{CS}$ is the generalized local Chern-Simons 3-form, i.e., the pairing of a 3-form and a 4-form

$$\begin{aligned}\mathcal{Q}_{CS} &= STr(\mathbf{A}\wedge\mathbf{dA}+\tfrac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A})\\ &= (STr(\mathcal{A}\wedge d\mathcal{A}+\tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}+k\mathcal{A}\wedge\mathcal{B}),\\ &\qquad STr(\mathcal{A}\wedge D\mathcal{B}+\mathcal{B}\wedge\mathcal{F}+k\mathcal{B}\wedge\mathcal{B})).\end{aligned} \tag{20}$$

Furthermore given the specific bilinear form of the supertrace for a superalgebra the super Chern-Simons terms can be written in components.

Thus, we have established the general formalism for a kind of $GTFTs$ and their supersymmetric generalizations via GDC. Significantly, it is straightforward to see that in the supersymmetry cases of the $GTFT$ the ordinary super-topological term in the action is also companied by the super-BF-type term and there is a relation between the super-BF-type term in the bulk of M and the super-Chern-Simons term on the boundary ∂M.

III. EINSTEIN-HILBERT ACTION AND YANG-MILLS ACTION AS CONSTRAINED BF GAUGE THEORIES

In this section, we first briefly recall how to reformulate the Einstein-Hilbert action together with corresponding topological term as constrained BF gauge theories in the framework of $GTFT$. Then we show that the Yang-Mills theory on the curved spacetime can also be reformulated as a $GTFT$ with constraints.

A. Einstein-Hilbert Action

For GR, it is well know that the gauge group is the homogeneous Lorentz group or its covering group $SL(2,C)$. Consider the $sl(2,C)$ algebra:

$$[J_{AB},J_{CD}] = \epsilon_{C(A}J_{B)D}+\epsilon_{D(A}J_{B)C}, \tag{21}$$

where $\epsilon_{C(A}J_{B)D}=\frac{1}{2}(\epsilon_{CA}J_{BD}+\epsilon_{CB}J_{AD})$. The local Minkowskian metric on the tangent space $\eta_{pq}=\mathrm{diag}(\eta_{(AB)(MN)})$ is given by

$$\eta_{(AB)(MN)} = \tfrac{1}{2}(\epsilon_{AM}\epsilon_{BN}+\epsilon_{AN}\epsilon_{BM}). \tag{22}$$

We introduce an $sl(2,C)$-valued generalized connection 1-form on the tangent bundle $P(M^4, SL(2,C))$ via GDC,

$$\mathbf{A} = (A^{AB}, B^{AB})J_{AB}, \tag{23}$$

where A^{AB} is the ordinary $sl(2,C)$-valued connection 1-form on the bundle and B^{AB} is an $SL(2,C)$-gauge covariant 2-form.

Given such a connection $\mathbf{A}$, the generalized curvature 2-form $\mathbf{F} = \mathbf{F}^p T_p = \mathbf{F}^{AB} J_{AB}$ can be given via GDC with the components

$$\mathbf{F}^{AB} = (F^{AB} + kB^{AB}, \quad DB^{AB}). \tag{24}$$

It does satisfies the Bianchi identity (5) via GDC.

A simple generalized topological Lagrangian 4-form in (6) of this curvature $\mathbf{F}$ is the second Chern class-like class as follows:

$$\begin{aligned} \mathcal{S}_{\mathrm{SL(2,C)}}[\mathbf{A}] &= \int_{M^4} \mathcal{L}_{\mathrm{SL(2,C)}} = \int_{M^4} \mathbf{F}^{AB} \wedge \mathbf{F}_{AB} + c.c. \\ &= \int_{M^4} (R^{AB} \wedge R_{AB} + 2kR^{AB} \wedge B_{AB} + k^2 B^{AB} \wedge B_{AB}) + c.c. \end{aligned} \tag{25}$$

It should be noted that first the second term in the above action is almost the same as the BF-type action in (1). Secondly, the 4-form Bianchi identity (the second term) in (5), $D^2 B^{AB} = 0$, looks identical to the Bianchi identity, $D^2(e^{AA'} \wedge e^B{}_{A'}) = 0$, of the first Cartan structure equation for ordinary torsion 2-form, $T^{AA'} := De^{AA'}$. Thus we may introduce 1-form fields $e^{AA'}$ to parameterize the tangent space of M, which is the coset $Sp(4)/SL(2,C)$, and take B^{AB} to be $l^{-2} e^{AA'} \wedge e^B{}_{A'}$ with l a dimensional constant [29].

The $sl(2,C)$-valued connection generalized 1-form now becomes [30]

$$\mathbf{A} = (A^{AB}, e^{AA'} \wedge e^B{}_{A'}) M_{AB} + c.c. \tag{26}$$

The action (25) becomes

$$\begin{aligned} &\mathcal{S}[A^{AB}, A^{A'B'}, e^{AA'}] = \int_{M^4} Tr(\mathbf{F} \wedge \mathbf{F}) + c.c. \\ &= \int_{M^4} (R^{AB} \wedge R_{AB} + 2kR^{AB} \wedge e^{AA'} \wedge e^B{}_{A'} + k^2 e^{AA'} \wedge e^B{}_{A'} \wedge e_A{}^{C'} \wedge e_{BC'} + c.c.), \end{aligned} \tag{27}$$

where the generalized curvature is given by

$$\begin{aligned} \mathbf{F} &= \mathbf{F}^{AB} M_{AB} + c.c. \\ \mathbf{F}^{AB} &= (F^{AB} + ke^{AA'} \wedge e^B{}_{A'}, D(e^{AA'} \wedge e^B{}_{A'})). \end{aligned} \tag{28}$$

Thus the action (25) is the Einstein-Hilbert action with the cosmological constant and a topological term. Namely, GR in the absence of matter may be formulated as a GTFT via GDC.

By varying with respect to A^{AB}, we obtain

$$D(e^{AA'} \wedge e^{B}{}_{A'}) = 0, \tag{29}$$

which gives the equation for torsion-free. While by varying with respect to $e^{AA'}$, we obtain

$$F^{AB} \wedge e_{B}{}^{A'} + k e^{AB'} \wedge e^{B}{}_{B'} \wedge e_{B}{}^{A'} + c.c. = 0, \tag{30}$$

which is the Einstein equation with a cosmological constant $\Lambda := k/l^2$.

Alternatively, we can consider adding a constraint on B^{AB} as in [10, 11]

$$\begin{aligned} \mathcal{S}[\mathbf{A}^p, \lambda_{AB}, e^{AA'}] &= \int_{M^4} \mathbf{F}^{AB} \wedge \mathbf{F}_{AB} + \lambda_{AB} \wedge (e^{AA'} \wedge e^{B}{}_{A'} - B^{AB}) + c.c. \\ &= \int_{M^4} F^{AB} \wedge F_{AB} + 2kF^{AB} \wedge B_{AB} + k^2 B^{AB} \wedge B_{AB} \\ &\quad + \lambda_{AB} \wedge (e^{AA'} \wedge e^{B}{}_{A'} - B^{AB}) + c.c., \end{aligned} \tag{31}$$

where λ_{AB} is the Lagrangian multiplier. The variational principle leads to the same equation with (30).

B. Yang-Mills Action on Curved Spacetime

Let us now consider Yang-Mills fields on the curved spacetime M^4. As is well known, the BF theory provides a strategy to write down Yang-Mills theory in a pure geometric formalism. Such a formulation has been studied at many places, for instance see[12, 13]. Quantizing this first-order formalism is supposed to give more insight into the non-local and non-perturbative features of $4D$ Yang-Mills theory. In this section we show that this formalism can be derived from the $GTFT$ via GDC as well.

Consider a semisimple gauge group G with generators T_p. We introduce a g-valued generalized connection 1-form as follows

$$\mathbf{A} = (A^p, B^p)T_p, \tag{32}$$

where A^p is the ordinary g-valued connection 1-form on the bundle $P(M^4, G)$ and B^p is a g-valued gauge covariant 2-form. Given such a generalized connection $\mathbf{A}$, the generalized

curvature $\mathbf{F} = \mathbf{F}^p T_p$ is given via GDC with components

$$\mathbf{F}^p = (F^p + kB^p, \quad DB^p). \tag{33}$$

A simple generalized Lagrangian 4-form using this connection $\mathbf{A}$ is

$$\begin{aligned} \mathcal{S}[\mathbf{A}] &= \int_{M^4} \mathbf{F}^p \wedge \mathbf{F}_p \\ &= \int_{M^4} (F^p \wedge F_p + 2kF^p \wedge B_p + k^2 B^p \wedge B_p). \end{aligned} \tag{34}$$

It is of the BF-type term with an ordinary second Chern-class.

Consider the constraint:

$$B^p = \phi^p_{AB} B^{AB}, \tag{35}$$

where $B^{AB} = e^{AC'} \wedge e^B{}_{C'}$. The action (34) becomes

$$\begin{aligned} &\mathcal{S}_{\rm YM}[A, \phi_{AB}, B^{AB}] = \int_{M^4} Tr(\mathbf{F} \wedge \mathbf{F}) \\ &= \int_{M^4} Tr[F \wedge F + 2kF \wedge \phi_{AB} B^{AB} + k^2 \phi_{AB} \phi_{CD} B^{AB} \wedge B^{CD}]. \end{aligned} \tag{36}$$

To see this action is actually the ordinary Yang-Mills theory in curved spacetime, we may vary this action with respect to ϕ_{AB}, then obtain the following equation of motion

$$kF \wedge B^{AB} + k^2 \phi_{CD} B^{AB} \wedge B^{CD} = 0, \tag{37}$$

which can be solved for ϕ_{CD}. Note that we can expand F to the basis of 2-forms $(B^{CD}, B^{C'D'})$[12],

$$F = F_{CD} B^{CD} + F_{C'D'} B^{C'D'}. \tag{38}$$

By comparing with the field equation for ϕ, we obtain

$$\phi_{AB} = -\frac{1}{k} F_{AB}. \tag{39}$$

It is easy to see that $-k\phi_{AB}$ is the self-dual part of the Yang-Mills curvature 2-form F^p and

$$B^p = \phi^p_{AB} B^{AB} = -\frac{1}{2k} (1 - *) F^p. \tag{40}$$

Inserting it into the action then yields

$$S_{YM} = \int_{M^4} \frac{1}{2} Tr[F \wedge *F] \tag{41}$$

that is just the usual Yang-Mills action on the curved spacetime M.

It should be mentioned that we obtain the Lagrangian of Yang-Mills theory simply by adding the *solutions* of constraints to the original actions of the GTFT. This is only for the sake of convenience. In first principle it is absolutely possible to add the constraints themselves to the actions and then solve these constraints.

For the Yang-Mills theory, we may add the constraints to the action:

$$\begin{aligned}\mathcal{S}_{\rm YM}[\mathcal{A}] &= \int_{M^4} \mathcal{L}_{\rm YM} = \int_{M^4} \mathbf{F}^p \wedge \mathbf{F}_p + \int_{M^4} \mathcal{L}_{\rm Constr.} \\ &= \int_{M^4} F^p \wedge F_p + 2kF^p \wedge B_p + k^2 B^p \wedge B_p - 2k^2 \phi^p_{AB} B_p \wedge B^{AB}. \end{aligned} \tag{42}$$

where ϕ^p_{AB} is the Lagrangian multiplier. Varying the action with respect to B_p yields the following equations of motion,

$$kB^p = k\phi^p_{AB} B^{AB} - F \tag{43}$$

Substituting it into the action we then obtain the action (36) up to a topological term and a sign difference ahead of the term $\phi_{AB}\phi_{CD}B^{AB} \wedge B^{CD}$, which does not affect any dynamics of Yang-Mills theory.

IV. N=1 CHIRAL SUPERGRAVITY AS CONSTRAINED TOPOLOGICAL FIELD THEORY

In next two sections we apply the general formalism for $GTFT$ via GDC to construct the first-order formalism of $N = 1, 2$ chiral supergravities.

As is know, the pure connection formulation of such kind of supergravity theories originally were presented by virtue of Ashtekar-Sen's variables in [14] and [15], respectively. The canonical quantization of $N = 1, 2$ supergravities based on this framework were considered in [15, 16] as well. Furthermore, Ezawa argued in [4] that $N = 1, 2$ supergravities can be cast into the form of topological field theories as for the case of GR. Here we will show that our general framework for $GTFT$ via GDC in the supersymmetric case provides a more direct manner to understand the topological origin of this formulation. In addition, our formalism also provides a convenient and compact way to construct such kind of supergravities. In fact, we find that by means of this formalism a class of topological field theories with free parameters may be set up while the usual known supergravities are the ones with the parameters specially chosen.

In this section we derive the Lagrangian of $N = 1$ chiral supergravity with the $Osp(1|2)$ algebra, which is the simplest supersymmetric extension of $su(2)$ algebra.

$$\begin{aligned}
[J_{AB}, J_{CD}] &= \epsilon_{C(A}M_{B)D} + \epsilon_{D(A}M_{B)C} \\
[J_{AB}, Q_C] &= \epsilon_{C(A}Q_{B)} \\
\{Q_A, Q_B\} &= aJ_{AB},
\end{aligned} \tag{44}$$

where a is a dimensionless constant to be fixed in order to give a reasonable gravity theory. Later we will find it is related to the cosmological constant as $a \sim -l^2\sqrt{-\Lambda}$. The super connection 1-form valued on this algebra is defined as

$$\mathcal{A} = A^{AB}J_{AB} + \psi^A Q_A. \tag{45}$$

In order to apply the GDC, we introduce a gauge covariant 2-form field

$$\mathcal{B} = B^{AB}J_{AB} + B^A Q_A. \tag{46}$$

Thus, we may define a generalized 1-form as

$$\mathbf{A} = (\mathcal{A}, \mathcal{B}) = (A^{AB}, B^{AB})J_{AB} + (\psi^A, B^A)Q_A. \tag{47}$$

Following the connection theory via GDC, the generalized curvature takes the form as

$$\begin{aligned}
\mathbf{F} &= \mathbf{dA} + \mathbf{A} \wedge \mathbf{A} \\
&= (F^{AB} + kB^{AB} - \frac{a}{2}\psi^A \wedge \psi^B, DB^{AB} - a\psi^A \wedge B^B)J_{AB} \\
&+ (F^A + kB^A, DB^A)Q_A,
\end{aligned} \tag{48}$$

where

$$F^{AB} = dA^{AB} + A^{AC} \wedge A_C{}^B \qquad F^A = d\psi^A + A^{AB} \wedge \psi_B. \tag{49}$$

For the action, we introduce the $GTFT$-type one (18)

$$S = \int_M STr(\mathbf{F} \wedge \mathbf{F}).$$

The 4-form components of its Lagrangian read as

$$\begin{aligned}
STr(\mathbf{F} \wedge \mathbf{F}) &= F^{AB} \wedge F_{AB} + 2kF^{AB} \wedge B_{AB} + k^2 B^{AB} \wedge B_{AB} \\
&+ \frac{a^2}{4}\psi^A \wedge \psi^B \wedge \psi_A \wedge \psi_B \\
&- aF^{AB} \wedge \psi_A \wedge \psi_B - akB^{AB} \wedge \psi_A \wedge \psi_B \\
&+ aF^A \wedge F_A + 2akF^A \wedge B_A + ak^2 B^A \wedge B_A,
\end{aligned} \tag{50}$$

where we have used bilinear relations

$$STrQ^AQ^B = a\epsilon^{AB}, \qquad STrJ_{AB}J^{CD} = \delta_A^C\delta_B^D. \tag{51}$$

Now we return to the action (50). This action can be further simplified by reading off the ordinary topological terms. Combining all the terms without B fields in the action we easily find that it is nothing but the ordinary supersymmetric version of the second Chern class $STr\mathcal{F}\wedge\mathcal{F}$ with supersymmetric curvature $\mathcal{F}$, which gives a total derivative term in the sense of ordinary differential calculus, i.e. $d\mathcal{L}_{CS}$[31]. If we read off all these ordinary topological terms without B fields, the remaining Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= 2kF^{AB}\wedge B_{AB} + k^2B^{AB}\wedge B_{AB} - akB^{AB}\wedge\psi_A\wedge\psi_B \\ &+ 2akF^A\wedge B_A + ak^2B^A\wedge B_A.\end{aligned} \tag{52}$$

Now it is straightforward to identify this action with a kind of the supersymmetric BF theories with the freedom of rescaling the coupling constants a, k and B fields. Therefore, our action (50) of the $GTFT$ type via GDC gives rise to a kind of supersymmetric BF theories up to an ordinary supersymmetric version of the second Chern class that is a totally derivative term. However, it should be noted that in our formalism both a and k are free parameters and not necessarily related to each other. The BF action given in [4] is obtained in the special case of $k = \frac{-a^2}{3}$ and $B'_A \sim aB_A$. In this sense we generalize the original formalism into a class of theories which of course should be topologically equivalent.

Let us now introduce the local degrees of freedom for both gravity and gravitino fields by constraining the B fields. This can be done by either adding new terms $\lambda_{(ABCD)}B^{AB}\wedge B^{CD}$ and $\lambda_{(ABC)}B^{AB}\wedge B^C$ into the action or simply plugging following solutions into the action,

$$B_{AB} := {e_A}^{A'}\wedge e_{A'B} \tag{53}$$

$$B_A := \frac{1}{a}{e_A}^{A'}\wedge\chi_{A'}, \tag{54}$$

where $\lambda_{(ABCD)}$ and $\lambda_{(ABC)}$ are Lagrangian multipliers with symmetrized indices. After that we find a kind of the Lagrangians with some free parameters as follows

$$\begin{aligned}\mathcal{L} &= 2k(F^{AB}\wedge {e_A}^{A'}\wedge e_{A'B} + \frac{k}{2}e^{AB'}\wedge {e^B}_{B'}\wedge {e_A}^{A'}\wedge e_{A'B} \\ &- \frac{a}{2}e^{AB'}\wedge {e^B}_{B'}\wedge\psi_A\wedge\psi_B \\ &+ F^A\wedge {e_A}^{A'}\wedge\chi_{A'} + \frac{k}{2a}e^{AB'}\wedge\chi_{B'}\wedge {e_A}^{A'}\wedge\chi_{A'}).\end{aligned} \tag{55}$$

Note that all these Lagrangians with arbitrary k and a are $N=1$ chiral supersymmetric.

In order to give the CDJ formalism of $N=1$ chiral supergravity [12], we need fix the constants k and a as

$$k = \frac{\Lambda}{3}, \qquad a = -\sqrt{-\Lambda}\ . \tag{56}$$

As a result, the final form of the known $N=1$ supergravity Lagrangian becomes

$$\begin{aligned} \mathcal{L}^{sugra} &:= \frac{\mathcal{L}}{2k} = (F^{AB} \wedge e_A{}^{A'} \wedge e_{A'B} + \frac{\Lambda}{6} e^{AB'} \wedge e^B{}_{B'} \wedge e_A{}^{A'} \wedge e_{A'B} \\ &+ \frac{\sqrt{-\Lambda}}{2} e^{AB'} \wedge e^B{}_{B'} \wedge \psi_A \wedge \psi_B \\ &+ F^A \wedge e_A{}^{A'} \wedge \chi_{A'} + \frac{\sqrt{-\Lambda}}{6} e^{AB'} \wedge \chi_{B'} \wedge e_A{}^{A'} \wedge \chi_{A'}) \end{aligned} \tag{57}$$

We note here Λ has to be negative due to the appearance of square root. It simply means we only have a Lagrangian with negative cosmological constant.

At the remainder of this section we briefly discuss the theory in the presence of boundary. It is well known that in this case we need some boundary terms so as to make the variation of the total action well defined. One option is adding such boundary terms by hand.

However, in our formalism one may have an alternative way to treat the boundary terms. Rather than adding any boundary action by hand as done in [10, 17, 18], we may induce an ordinary super Chern-Simons action on the boundary from the topological term $\frac{1}{2k} STr\mathcal{F}\wedge\mathcal{F}$ in the bulk. As we noticed, it is nothing but the topological term

$$STr(\mathcal{F} \wedge \mathcal{F}) = d\mathcal{L}_{CS}, \tag{58}$$

where

$$\begin{aligned} \mathcal{L}_{CS} &= STr(\mathcal{A} \wedge d\mathcal{A} + \tfrac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \\ &= A^{AB} \wedge dA_{AB} + \frac{2}{3} A^A{}_C \wedge A^C{}_B \wedge A^B{}_A - a\psi^A \wedge D\psi_A. \end{aligned} \tag{59}$$

Thus in the presence of boundary, the total action of supergravity is

$$S \equiv \frac{S}{2k} = \int_M \mathcal{L}^{sugra} + \frac{1}{2k}\int_{\partial M} (A^{AB} \wedge dA_{AB} + \frac{2}{3} A^A{}_C \wedge A^C{}_B \wedge A^B{}_A - a\psi^A \wedge D\psi_A). \tag{60}$$

Interestingly enough, from (60) we find the coefficient of the second Chern class $\frac{\kappa}{8\pi}$ should be fixed by the cosmological constant as

$$\frac{\kappa}{8\pi} = \frac{1}{2k} = \frac{3}{2\Lambda}. \tag{61}$$

Furthermore, to make the variation of the total action well defined, we still need impose the boundary condition, which should be

$$STr\delta\mathcal{A}\wedge\mathcal{B}=\frac{3}{\Lambda}STr\delta\mathcal{A}\wedge\mathcal{F}. \tag{62}$$

V. N=2 CHIRAL SUPERGRAVITY AS CONSTRAINED TOPOLOGICAL FIELD THEORY

In this section we demonstrate that our formalism for the $GTFT$ via GDC can be extended to $N=2$ superalgebra in almost the same manner as in previous sections and thus applicable to the construction of a sort of $N=2$ supergravities with some free parameters. The known $N=2$ supergravity is the one with the parameters specially chosen.

The Lie super-algebra corresponding to $N=2$ chiral supergravity is $Osp(2|2)$. It is composed of bosonic generators J_{AB},Q which span the algebra $sp(2)\times o(2)$, and fermionic generators $Q^i_A(i=1,2)$. The algebraic relations read as

$$\begin{aligned}
[J_{AB},J_{CD}] &= \epsilon_{C(A}M_{B)D}+\epsilon_{D(A}M_{B)C}\\
\left[J_{AB},Q^i_C\right] &= \epsilon_{C(A}Q^i_{B)}\\
\left\{Q^i_A,Q^j_B\right\} &= aJ_{AB}\delta^{ij}+\epsilon_{AB}\epsilon^{ij}Q\\
\left[Q,Q^i_A\right] &= \frac{1}{2}a\epsilon^{ij}Q_{Aj}\\
[Q,Q] &= [Q,J_{AB}]=0,
\end{aligned} \tag{63}$$

where we introduce the matrix ϵ^{ij}

$$\epsilon^{ij}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{64}$$

To check the closure of this superalgebra it is useful to note the following identity

$$\epsilon^{ij}\epsilon^{km}=\delta^{jk}\delta^{im}-\delta^{ik}\delta^{jm}. \tag{65}$$

The super connection 1-form now is defined as

$$\mathcal{A}=A^{AB}J_{AB}+\psi^A_iQ^i_A+AQ, \tag{66}$$

where A is a $U(1)$ connection. In parallel we introduce a gauge covariant 2-form field

$$\mathcal{B}=B^{AB}J_{AB}+B^A_iQ^i_A+BQ. \tag{67}$$

The generalized connection 1-form is defined as the pairing

$$\mathbf{A} = (\mathcal{A}, \mathcal{B}) = (A^{AB}, B^{AB})J_{AB} + (\psi_i^A, B_i^A)Q_A^i + (A, B)Q. \tag{68}$$

Thus the generalized curvature 2-form are given by

$$\begin{aligned}
\mathbf{F} &= \mathbf{dA} + \mathbf{A} \wedge \mathbf{A} \\
&= (F^{AB} + kB^{AB} - \frac{a}{2}\psi^{iA} \wedge \psi_i^B, DB^{AB} - a\psi^{iA} \wedge B_i^B)J_{AB} \\
&+ (F_i^A + kB_i^A + \frac{1}{2}a\epsilon_{ij}A \wedge \psi^{jA}, DB_i^A + a\epsilon_{ij}A \wedge B^{jA})Q_A^i \\
&+ (dA + kB + \frac{1}{2}\epsilon^{ij}\psi_i^A \wedge \psi_{jA}, dB + \epsilon^{ij}\psi_i^A \wedge B_{jA})Q.
\end{aligned} \tag{69}$$

It satisfies the Bianchi identity with respect to $\mathbf{d}$.

The generalized topological action can be constructed with the use of this supercurvature 2-form as

$$S = \int_M STr\mathbf{F} \wedge \mathbf{F} \tag{70}$$

where

$$STrQ_A^iQ_B^j = a\epsilon_{AB}\delta^{ij}, \quad STrJ_{AB}J^{CD} = \delta_A^C\delta_B^D, \quad STrQQ = a^2. \tag{71}$$

More explicitly its 4-form components read as

$$\begin{aligned}
STr\mathbf{F} \wedge \mathbf{F} &= F^{AB} \wedge F_{AB} + 2kF^{AB} \wedge B_{AB} + k^2B^{AB} \wedge B_{AB} \\
&+ \frac{a^2}{4}\psi^{iA} \wedge \psi_i^B \wedge \psi_A^j \wedge \psi_{jB} \\
&- aF^{AB} \wedge \psi_A^i \wedge \psi_{iB} - akB^{AB} \wedge \psi_A^i \wedge \psi_{iB} \\
&+ a[F^{iA} \wedge F_{iA} + 2kF^{iA} \wedge B_{iA} + k^2B^{iA} \wedge B_{iA} \\
&+ akB^{iA} \wedge A \wedge \epsilon_{ij}\psi_A^j + \frac{a^2}{4}A \wedge \psi^{iA} \wedge A \wedge \psi_{iA} + aF^{iA} \wedge A \wedge \epsilon_{ij}\psi_A^j] \\
&+ a^2[F \wedge F + k^2B \wedge B + 2kF \wedge B + k\epsilon^{ij}\psi_i^A \wedge \psi_{jA} \wedge B \\
&+ F \wedge \epsilon^{ij}\psi_i^A \wedge \psi_{jA} + \frac{1}{4}\epsilon^{ij}\psi_i^A \wedge \psi_{jA} \wedge \epsilon^{mn}\psi_m^B \wedge \psi_{nB}],
\end{aligned} \tag{72}$$

Again, we separate all the terms without B fields and combine them together, leading to an ordinary topological term $STr\mathcal{F} \wedge \mathcal{F}$ where $\mathcal{F}$ is the $Osp(2|2)$-valued curvature 2-form. Reading off this topological term, we obtain the following BF type Lagrangians with free parameters k, a as follows:

$$\mathcal{L} = 2k(F^{AB} \wedge B_{AB} + \frac{k}{2}B^{AB} \wedge B_{AB} - \frac{a}{2}B^{AB} \wedge \psi_A^i \wedge \psi_{iB}$$

$$
\begin{aligned}
&+\ aF^{iA} \wedge B_{iA} + \frac{ak}{2} B^{iA} \wedge B_{iA} + \frac{a^2}{2} B^{iA} \wedge A \wedge \epsilon_{ij} \psi^j_A \\
&+\ \frac{ka^2}{2} B \wedge B + a^2 dA \wedge B + \frac{a^2}{2} B \wedge \epsilon^{ij} \psi^A_i \wedge \psi_{jA}).
\end{aligned} \tag{73}
$$

We are free to rescale the B fields as $B^A_i \sim \frac{1}{a} B^A_i$, $B \sim \frac{1}{a^2} B$. Furthermore, to identify our Lagrangian with that in [4], we redefine the coupling constants

$$
a = 2g, \quad k = 2g^2 = \frac{-\Lambda}{3}. \tag{74}
$$

As a result,

$$
\begin{aligned}
\frac{\mathcal{L}}{2k} &= F^{AB} \wedge B_{AB} + g^2 B^{AB} \wedge B_{AB} - g B^{AB} \wedge \psi^i_A \wedge \psi_{iB} \\
&+\ F^{iA} \wedge B_{iA} + \frac{g}{2} B^{iA} \wedge B_{iA} + g B^{iA} \wedge A \wedge \epsilon_{ij} \psi^j_A \\
&+\ \frac{1}{4} B \wedge B + \hat{F} \wedge B.
\end{aligned} \tag{75}
$$

where

$$
\hat{F} = dA + \frac{1}{2} \epsilon^{ij} \psi^A_i \wedge \psi_{jA} \tag{76}
$$

The supergravity Lagrangian is obtained by plugging the following constraints into (75):

$$
B_{AB} = e_A{}^{A'} \wedge e_{BA'} \qquad B_{Ai} = e_A{}^{A'} \wedge \chi_{iA'}, \tag{77}
$$

and

$$
e_A{}^{A'} \wedge e_{BA'} \wedge B = e_A{}^{A'} \wedge \chi_{iA'} \wedge e_B{}^{B'} \wedge \chi^i_{B'}, \tag{78}
$$

The subtlety that the last constraint indeed gives the Maxwell term of $U(1)$ field can be understood following our discussion in Yang-Mills section. Namely after plugging the constraints into the action of BF theory, we obtain the terms involving the auxiliary field B in action (75) as

$$
\mathcal{L}_{U(1)} = \frac{1}{4} B \wedge B + \hat{F} \wedge B + \phi_{AB} (B^{AB} \wedge B - e^{A'}_A \wedge \chi_{iA'} \wedge e^{B'}_B \wedge \chi^i_{B'}). \tag{79}
$$

Varying with B yields the equation of motion

$$
\frac{1}{2} B = -\hat{F} - \phi_{AB} B^{AB}. \tag{80}
$$

Substituting this solution back to the action immediately leads to the chiral action of $N = 2$ supergravity given in [15].

Note that this known chiral action of $N = 2$ supergravity is just the special case with the chosen parameters (74). In fact, as in the case of $N = 1$ chiral supergravities, a kind of $N = 2$ chiral supergravities with free parameters should also be set up in general in our formalism of the $GTFT$ with constraints via GDC.

VI. CONCLUDING REMARKS

In this paper, we have further explored our approach to the connection theory and $GTFT$ with constraints via the GDC [6] to both the Einstein-Hilbert action in GR and Yang-Mills action on curved spacetime. We have also extended it as a general formalism including the supersymmetric cases. In particular we have constructed $N = 1, 2$ supersymmetric BF-theories with free parameters as well as chiral supergravities as the parameters are specially chosen. The common properties of such kinds of $GTFTs$ are of the BF-type companying with the ordinary second Chern-class in the bulk and the Chern-Simons term on the boundary. This not only has recovered the deep relation between the BF-type TFT and the Chern-Simons one, but also has combined them together in the bulk or on the boundary, respectively, rather than added by hand as has been done in literatures. In all these cases, Ashtekar-Sen's variables have been employed. This is quite convenient, although it is not necessary.

It should be noticed that such a general formalism works only for the (super)gravity theories with non-zero cosmological constants. This is much similar to the case that Chern-Simons states as the topological solutions to quantum GR exist only in the case of non-zero cosmological constants. This similarity is sharpen if we observe the total action (60) where the coefficients of Chern-Simons action on the boundary has to be fixed inversely proportional to the cosmological constants as well as Chern-Simons states in quantum gravity. As we know the Chern-Simons theory on the boundary plays the key role in the holographical formulation of GR at quantum mechanical level [10, 17], it is reasonable to conjecture that the holographical interpretation of quantum gravity make sense only for theories with non-zero cosmological constants.

It is worthwhile to remind that the basic properties of four dimensional BF theory have previously been studied in many references, for instance see [19–21]. Recently it turns out that BF formulation of GR plays a crucial role in the quantization of gravitational fields as

well. Some of its important applications may be illustrated as follows.

First one is to study the evolution of spacetime at quantum mechanical level. Originally Crane and Yetter proposed that using BF theory the algebraic method in topological quantum field theory can be applied to construct quantum geometry in the form of a discrete model on a triangulated 4-manifold [22, 23]. This idea is quite similar to the Ponzano-Regge state sum model for three dimensional gravity, but one significant difference here is that gravity has the local degrees of freedom in four dimensions, which requires imposing some additional constraints to BF Lagrangian. After the appearance of spin networks in loop quantum gravity, these ideas ware quickly applied to the construction of state sum model [24] and spin foams [25, 26].

The second application is to construct holographic formulation of non-perturbative quantum gravity and supergravity[10, 17], motivated by the original work of 't Hooft and Susskind, who roughly speaking conjecture that in quantum gravity the state space describing physics in a region with finite volume should be finite dimensional. The basic idea here is to study the quantization of gravity in the presence of finite boundary. It turns out that BF formulation of gravity in this context provides not only an ideal framework for imposing the appropriate boundary conditions on spacetime, but also the key mathematical structure to construct the boundary theory of quantum gravity.

The third application is the study of isolated horizons[18]. Given a BF formulation of gravity in the bulk, the strategy of describing quantum gravity on the isolated horizons can be carried out closely following the idea in [10], while the main different ingredient here is the treatment of boundary conditions.

It is needless to say that in all these applications our general formalism may play some essential role when we want to extend above considerations to supergravity theories.

In our previous paper [6] and this paper, the 4-dimensional manifolds are concerned. In principle, our approach may apply to other dimensions. Although it seems that the action of GR in Ashtekar-Sen variables may not be applicable for the manifolds rather than four dimensions, some kinds of BF actions may appear together with topological one as the candidates for the $GTFT$. On the other hand, as was mentioned in the context, there may exist a kind of decent relations among the topological terms on different dimensions since in the action of $GTFT$ the topological terms of different forms appear already via GDC.

Furthermore we notice that the concept of a generalized p-form discussed in our paper

is only a special case of a broader generalization in which generalized p-forms may be constructed by $(n+1)$-tuples of ordinary forms [7, 8]. It is of particular interest to investigate the physical application to higher dimensional supergravity theories where generalized p-form connections are allowed to associate to $(p-1)$-branes.

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Appendix: Generalized Differential Calculus

A generalized p-form [8, 27], $\overset{p}{\mathbf{a}}$, is defined to be an ordered pair of an ordinary p-form $\overset{p}{\alpha}$ and an ordinary $(p+1)$-form $\overset{p+1}{\alpha}$ on an n-dimensional manifold M, that is

$$\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha}) \in \Lambda^p \times \Lambda^{p+1}, \tag{81}$$

where $-1 \leq p \leq n$. The minus one-form is defined to be an ordered pair

$$\overset{-1}{\mathbf{a}} \equiv (0, \overset{0}{\alpha}), \tag{82}$$

where $\overset{0}{\alpha}$ is a function on M. The product and derivatives are defined by

$$\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} \equiv (\overset{p}{\alpha} \wedge \overset{q}{\beta}, \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta}), \tag{83}$$

$$\mathbf{d} \overset{p}{\mathbf{a}} \equiv (d \overset{p}{\alpha} + (-1)^{p+1} k \overset{p+1}{\alpha}, d \overset{p+1}{\alpha}), \tag{84}$$

where k is a constant. These exterior products and derivatives of generalized forms satisfy the standard rules of exterior algebra

$$\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} = (-1)^{pq} \overset{q}{\mathbf{b}} \wedge \overset{p}{\mathbf{a}}, \tag{85}$$

$$\mathbf{d}(\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}}) = \mathbf{d} \overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} + (-1)^p \overset{p}{\mathbf{a}} \wedge \mathbf{d} \overset{q}{\mathbf{b}}, \tag{86}$$

and $\mathbf{d}^2 = 0$.

For a generalized p-form $\overset{p}{\mathbf{a}}\,(\overset{p}{\alpha}, \overset{p+1}{\alpha})$, the integration on M^p can be defined as usual by

$$\int_{M^p} \overset{p}{\mathbf{a}} = \int_{M^p} (\overset{p}{\alpha}, \overset{p+1}{\alpha}) = \int_{M^p} \overset{p}{\alpha} \,. \tag{87}$$

The generalized connection and curvature on $P(M,G)$ have been introduced and the generalized Chern-Weil homomorphism for generalized curvature invariant polynomials in any even dimensional manifolds have also been established. But, their topological meaning should be as same as before. This can be understood by the fact that for $\mathbf{d}$, the cohomology is trivial. We show this property as follows. Given a generalized p-form which is closed in the context of GDC, namely

$$\mathbf{d}(\overset{p}{\alpha}, \overset{p+1}{\beta}) = (d\,\overset{p}{\alpha} + (-1)^{p+1} k\,\overset{p+1}{\beta},\; d\,\overset{p+1}{\beta}) = 0, \tag{88}$$

then we have

$$d\,\overset{p}{\alpha} + (-1)^{p+1} k\,\overset{p+1}{\beta} = 0, \quad d\,\overset{p+1}{\beta} = 0 \tag{89}$$

for any non-zero constant k. Thus,

$$\overset{p+1}{\beta} = (-1)^p k^{-1} d\,\overset{p}{\alpha} \,. \tag{90}$$

It follows

$$(\overset{p}{\alpha},\; (-1)^p k^{-1} d\,\overset{p}{\alpha}) = \mathbf{d}(0,\; (-1)^p k^{-1}\,\overset{p}{\alpha}).$$

Namely, the closed generalized p-form is always exact. Therefore the cohomology is trivial.

This property implies that the ordinary BF theory combined with the ordinary second Chern class may share the same topological meaning with the ordinary second Chern class. The generalized topological field theory just provides a framework for the combination of these two sorts of theories. Then it is quite natural to see that for an ordinary BF theory in the bulk, we may have a Chern-Simons action induced on the boundary in this context.

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[29] We will ignore this dimensional constant as well as the Newton constant through this paper for convenience. But since $e^{AA'}$ is dimensionless while ψ^A has the dimension $l^{-1/2}$, it's easy to recover all these dimensional factors in the Lagrangian.

[30] The formulation can be written with purely unprimed spinors by defining spinor 1-forms $\varphi^A = e^{A0'}$ and $\chi^A = e^{A1'}$ [28]. In terms of these spinor 1-forms, the purely unprimed $sl(2,C)$-valued generalized connection 1-form is $\mathcal{A}^+ = \left(\omega^{AB}, \;\; (2/l^2)\; \chi^{(A} \wedge \varphi^{B)}\right) M_{AB}$.

[31] Of course among these terms we may immediately delete the term with four ψs due to the fact $\psi^A \wedge \psi_A = 0$, but it would be better to keep in mind that such terms with four ψs do not necessarily vanish in $N > 1$ supergravities since spinors are specified by more indices. One example appearing in next section is that $\psi_i^A \wedge \psi_{jA} = 0$ only if i, j is symmetric.